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ELECTRICAL ANALOGIES FOR STIFFENED SHELLS WITH FLEXIBLE RINGS

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SUMMARY

Structural theory and analogous electrical circuits are developed for stiffened shells with flexible rings. By assumption, the forces that a shell (consisting of stringers and skin) and a ring can exert on each other are directed along their common line of intersection. As a consequence the shell and the rings can be treated separately.

First an electrical analogy is developed for a circular shell with a straight axis and variable radius. This analogy is extended to non-circular cylinders. Next an electrical analogy is derived for rings with variable radii of curvature; a simplified circuit for circular rings is also presented. The simplifications that occur when the rings are assumed to be rigid are discussed. Finally results are given for two sample problems solved on an analog computer. The second problem concerns a cantilever conical shell and illustrates the manner in which the shell and ring circuits are interconnected.

INTRODUCTION

The type of shell considered in this paper has an elongated shape and consists of a thin skin supported by stringers and rings. In analyzing such shells it is the nearly universal practice to replace the elastic supporting rings by rigid bulkheads in order to simplify the analysis. This will not be done in this paper.

The means of analysis to be used in this paper is an electric analog computer of the "direct analogy" type. Any complicated system, if it is to be analyzed on such a computer, must have its equations formulated in a very special way. Essentially one seeks for laws of equivalence between the system being analyzed and a lumped constant electrical network. The basic laws of equivalence between the equations of elasticity and the equations of an electrical circuit are well known. In fact there are two alternative sets of laws depending on whether force is made analogous to current or to voltage. If the former alternative is chosen the laws of equivalence are: Force is analogous to current, displacement is analogous

to voltage, Hooke's law is analogous to Ohm's law, equations of equilibrium are analogous to Kirchhoff's law for the sum of currents entering a node, and the equations concerning the compatibility of strains are analogous to Kirchhoff's law for the voltages around a loop.

However much comfort these basic laws of equivalence may give they are usually insufficient to determine the form of a lumped-constant electrical network that is analogous to a given structure. For one thing elastic structures are continuous rather than "lumped," and some means must be found for replacing the given continuous elastic structure by an idealized lumped one before an electrical analogy can be found. This "lumping" consists either of replacing the differential equations governing the structure by finite-difference equations, or of employing other devices such as concentrating normal-stress-carrying area into equivalent flanges and shear-carrying area into equivalent panels. In the analysis of stiffened structures this latter approach is reinforced by the fact that much of the structure is in fact so concentrated. In this paper both of the methods mentioned will be used.

In deriving an electrical analogy for an elastic structure an effort should be made to preserve a one-to-one correspondence between the properties of the electrical circuit and the properties of the idealized structure. This correspondence means, for example, that the current in resistor A is equal to the force in flange A' multiplied by a scale factor, or that the voltage at node B is equal to the vertical displacement at panel point B' multiplied by a scale factor. If such correspondences are preserved, the analog computer can be made a useful tool for designing as well as for analyzing structures. If a change in the cross-sectional area of a single flange corresponds to changing the value of a single resistor and if currents can be easily and directly converted into internal forces, then design changes can be made very rapidly and their effects instantly determined while the problem is set up on the analog computer. In the present paper these correspondences are rigidly preserved.

Another advantage of the close one-to-one correspondence of the electrical analogy and the idealized structure is that it enables the structural engineer, who is usually uninstructed in electric-circuit theory, to understand the operation of the analog computer and to use it himself after a period of indoctrination. It has even been found that structural engineers may be aided in their understanding of structures by using some of the concepts of electric-circuit theory. This naturally applies, a fortiori, to the electrical engineer.

The present paper is intended as a step in the development of an analog-computer method that is generally applicable to the solution of aircraft structural problems. At present, analogies exist in the technical literature for beams, frameworks, flat sheets, the bending of

plates, and the bending of platelike multicell shells. Structures combining components of the above types can be analyzed by combining their electrical analogies. Consequently it is at present possible to analyze a great many practical aircraft structures.

Some of the previous papers that have a direct bearing on the subject of the present paper should be mentioned.

In 1944 Kron published a paper containing electrical analogies for the general three-dimensional elastic-field problem and, as subcases, analogies for the plane-stress and plane-strain problems (ref. 1). In a companion paper, Carter worked the plane-stress problem for a deep cantilever beam (ref. 2).

More recently analogies have been developed for thin multicell shells having a horizontal plane of symmetry by using an equivalent plate theory (ref. 3).

In 1951 Goran published a paper containing an electrical analogy for stiffened elastic shells (ref. 4). The shells were assumed to be conical and to be supported by rigid bulkheads. Although the stringers were not assumed to be parallel, the panels were assumed to be nearly rectangular in shape. Goran used a minimum energy principle in deriving the equations from which he developed the electrical analogy, in contrast with the method of difference equations used in this paper.

The present investigation was conducted at the California Institute of Technology and has been made available to the National Advisory Committee for Aeronautics for publication because of its general interest.

SYMBOLS

A_s	cross-sectional area of stringer
E	Young's modulus
f_n	external load normal to ring per radian of ϕ
f_t	external load tangential to ring (or skin) per radian of ϕ
F_n	shear force in ring
F_s, F_s'	forces in stringer
F_{st}, F_{st}'	tangential forces in panel parallel to t

F_t	axial force in ring
F_{ts}, F_{ts}'	tangential forces in panel parallel to s
G	shear modulus
h	thickness of skin
I	moment of inertia of ring cross section
M	bending moment in ring
P	applied vertical load
r	radius of circular shell; radius of curvature of ring; distance to point in rigid ring
s	coordinate parallel to stringer
t	coordinate perpendicular to stringer and parallel to ring
u_n	displacement normal to axis of ring
u_s	displacement parallel to s
u_t	displacement parallel to t
$\bar{u}_t$	coordinate related to u_t by transformation
Y	horizontal displacement of rigid bulkhead
y	horizontal direction
Z	vertical displacement of rigid bulkhead
z	vertical direction
α	angle between two adjacent stringers; angle between tangential displacement and line to a point in a rigid bulkhead
γ	shear strain of panel
δ	angle by which direction of s-axis is changed because of translation in t-direction
$\Delta_s, \Delta_t, \Delta_\phi$	difference operators in s-, t-, and ϕ-directions
$\Delta s, \Delta t, \Delta\phi$	increments in s, t, and ϕ

$\Delta\phi' = 2 \sin (\Delta\phi/2)$

ϕ angle between horizontal line and normal to center line of undisplaced ring

ψ angle between stringer and axis of shell

θ rotation about axis of shell, usually in ring

DERIVATION OF AN ANALOGY FOR A CIRCULAR NONCYLINDRICAL STIFFENED SHELL

A sketch of a circular noncylindrical shell is shown in figure 1(a). This shell consists of circular elastic rings to which stringers and a thin skin covering are attached. The rings are spaced a finite distance apart in planes perpendicular to the axis of the shell, which is assumed to be straight. The stringers lie in planes perpendicular to the rings and are assumed to carry axial forces only. The skin is divided into panels by the intersection of the stringers and rings; these panels are assumed to carry shearing forces only, the effective normal-stress-carrying area of the skin having been included in the cross-sectional areas of the stringers and rings. The radius of the shell may vary in any manner along the axis of the shell, but the radius of curvature of the stringers is assumed to be large compared with the radius of the shell. The angle between the axis of the stringers and the axis of the shell need not be small provided that the rings and stringers are spaced close enough together so that the shear panels are approximately rectangular in shape.

The stringers and skin will be treated separately from the rings. For the analysis of the stringers and skin it will be shown that the rings can be represented by tangential external forces applied to the skin along the lines of intersection of the skin and the rings.

The equilibrium and force-displacement equations will be derived for the skin and stringers and an electrical circuit satisfying these equations will be constructed. Then in the next section the equations for an elastic ring will be written and a circuit satisfying these equations will be constructed. As a final step the two circuits will be connected together to give the electrical analogy for the whole shell. Orthogonal coordinates in the surface of the shell parallel and perpendicular to the stringers as shown in figure 1(a) will be employed in the analysis of the skin and stringers. An enlarged portion of the shell is shown in figure 1(b). This figure shows the points where displacements parallel to the axis of the stringers u_s and displacements parallel to the axis of the rings u_t are defined.

Figure 2 shows a portion of the skin between two adjacent rings with its midpoint on a stringer. The total axial forces carried by the stringer at points where the stringer passes over two adjacent rings are F_s and F_s'. The total tangential forces acting in the s-direction on sections passing through the centers of two adjacent shear panels are F_{ts} and F_{ts}'.

The force in the stringer is continuous at the point where it passes over a ring because of the following assumptions:

(a) The ring can exert only forces which lie in its own plane
(b) The stringer cannot support bending loads
(c) The change in direction of the stringer at the point where it passes over a ring is negligibly small compared with the curvature of the ring.[1]

Consequently the ring can exert forces only in the t-direction, and these forces can be treated as applied forces in the analysis of the skin and stringers.

In figure 2, $\alpha/2$ is the angle between the stringer and the line of action of the shear force F_{ts}'. This angle is approximately equal to $(1/2)\psi\,\Delta\phi$ where ψ is the angle between the axis of the shell and the axis of the stringer and $\Delta\phi$ is equal to the angle subtended by a segment of ring between two adjacent stringers. The cosine of the product of these angles will be assumed to be equal to 1. Hence the equilibrium equation for forces in the s-direction is:

$$F_s' - F_s + F_{ts}' - F_{ts} = 0 \tag{1}$$

Using difference-equation notation, equation (1) can be written

$$\Delta_s F_s + \Delta_t F_{ts} = 0 \tag{2}$$

This notation will be used in the remainder of this paper.

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[1]It is not difficult to relax this assumption to permit an appreciable change in direction of the stringer and a consequent radial load on the ring. The resulting electrical circuit contains an additional transformer at each point of intersection of stringer and ring.

Figure 3 shows a portion of the skin between two stringers with its midpoint on a ring. The total tangential forces acting in the t-direction on sections passing through the centers of two adjacent shear panels are F_{st} and F_{st}'. The tangential force exerted by the ring on the skin per radian of ϕ is f_t. The lines of action of the forces F_{ts} and F_{ts}' intersect the axis of the shell, so that the equation of equilibrium for moments about this axis is

$$\Delta_s(rF_{st}) - r\Delta\phi f_t = 0 \tag{3}$$

The total forces acting on perpendicular planes passing through the center of a shear panel are F_{ts} and F_{st}. From equation (3) it is seen that the shear stress cannot be uniform in the s-direction across the surface of a panel if r is not constant. Hence the assumption that the panel carries only shear stresses is incorrect. The secondary normal stresses required by equation (3) are ignored.

It will be assumed that the variation of shear stress is linear across the surface of the panel, so that the value at the center of the panel is equal to the average along a line in either the s- or t-direction. Then a relationship between F_{ts} and F_{st} may be obtained from the equilibrium equation of a small element at the center of the panel (see fig. 4):

$$F_{ts} \frac{\delta s}{\Delta s} \delta t = F_{st} \frac{\delta t}{\Delta t} \delta s \tag{4}$$

$$F_{ts} = F_{st} \frac{\Delta s}{\Delta t} \tag{5}$$

The equilibrium of the portion of the shell shown in figure 3 for forces in a direction parallel to t at the center of the section may be demonstrated, if desired, by means of equations (3) and (5).

The force-displacement equation for the stringers is quite easily written. The axial displacement of the stringer u_s is defined at the midpoints between adjacent rings as shown in figure 1(b) while the axial force (positive for tension) is defined at points where the stringer passes over the rings as shown in figure 2. Assuming the variation in axial force to be linear between adjacent points where u_s is defined, from Hooke's law

$$F_s = \frac{EA_s}{\Delta s}\left(\Delta_s u_s\right) \tag{6}$$

where A_s is the cross-sectional area of the stringer.

The relationship between shear force and displacements for the shear panels is less easily written because the relationship between shear strain and the displacements is complicated. The displacements in the s- and t-directions are defined at the midpoints of the sides of the shear panel as shown in figure 1(b). The shear strain of the panel is defined as the distortion of the angle between two lines passing through the midpoints of the sides. In computing this angle care must be taken to eliminate apparent distortion due to rigid body rotation about the axis of the shell. From figure 5, the shear strain

$$\gamma = \gamma_1 + \gamma_2 \tag{7}$$

where

$$\gamma_1 = \frac{\Delta_t u_s}{\Delta t} \tag{8}$$

$$\gamma_2 = \frac{\Delta_s u_t}{\Delta s} - \delta \tag{9}$$

and δ is the angle by which the direction of the s-axis, drawn through the center of the panel, has been changed because of translation parallel to the t-axis. For small displacements this angle is

$$\delta = \frac{u_t}{r} \sin \psi \tag{10}$$

where ψ is the angle between the axis of the shell and the s-axis. From figure 2 it can be seen that

$$\frac{\partial r}{\partial s} = \sin \psi \tag{11}$$

Combining equations (9), (10), and (11),

$$\gamma_2 = \frac{\Delta_s u_t}{\Delta s} - \frac{u_t}{r}\frac{\partial r}{\partial s} \tag{12}$$

Assume that the first term on the right can be replaced by the equivalent partial derivative evaluated at the center of the panel

$$\gamma_2 = \frac{\partial u_t}{\partial s} - \frac{u_t}{r}\frac{\partial r}{\partial s} = r\frac{\partial}{\partial s}\left(\frac{u_t}{r}\right) \tag{13}$$

Replace this equation by its finite difference equivalent and obtain from equations (7), (8), and (13):

$$\gamma = \frac{\Delta_t(u_s)}{\Delta t} + \frac{r}{\Delta s}\Delta_s\left(\frac{u_t}{r}\right) \tag{14}$$

The shearing strain is related to the total tangential force acting on the panel by the following equation

$$\gamma = \frac{1}{Gh}\frac{F_{ts}}{\Delta s} \tag{15}$$

where h is the thickness of the skin. Hence the force-displacement equation for the panel is

$$F_{ts} = Gh\frac{\Delta s}{\Delta t}\left[\Delta_t(u_s) + \frac{r\Delta t}{\Delta s}\Delta_s\left(\frac{u_t}{r}\right)\right] \tag{16}$$

The equations that are essential for the construction of an electrical analogy are summarized below. The equilibrium equations are:

$$\Delta_s F_s + \Delta_t F_{ts} = 0 \tag{17a}$$

$$\Delta_s(rF_{st}) - \Delta t f_t = 0 \tag{17b}$$

$$F_{ts} = F_{st}\frac{\Delta s}{\Delta t} \tag{17c}$$

The force-displacement equations are:

$$F_s = \frac{EA_s}{\Delta s}\left(\Delta_s u_s\right) \tag{17d}$$

$$F_{ts} = Gh\,\frac{\Delta s}{\Delta t}\left[\Delta_t(u_s) + \frac{r\Delta t}{\Delta s}\,\Delta_s\left(\frac{u_t}{r}\right)\right] \tag{17e}$$

These equations have been derived by assuming the normal-stress-carrying area of the shell to be concentrated in stringers. They can also be derived from the differential equations for a membrane shell of revolution (ref. 5) by replacing differential operators by finite difference operators. This is an important fact because it extends the applicability of the equations to certain unstiffened shells.

In the electrical analogy forces are analogous to currents and displacements are analogous to voltages. The complete circuit is shown in figure 6. This circuit consists of two separate parts. In one part the voltages to ground are the displacements u_s while in the other part the voltages to ground are the rotations u_t/r. The two circuits are coupled together by means of ideal transformers. Transformer coils which are coupled together are indicated by circled numbers. Points at which each one of the above equations are satisfied are indicated by letters in the circuit. Equation (17a) is satisfied by the currents entering a node of the u_s circuit. Equation (17b) is satisfied by the currents entering a node of the u_t circuit. Equation (17c) is satisfied by the currents flowing in the windings of a transformer whose turns ratio is $r\Delta t/\Delta s$. Equation (17d) is satisfied by a resistor whose value in ohms is $\Delta s/EA_s$. In equation (17e) the increment in u_s in the t-direction is added to a fraction of the increment in u_t/r in the s-direction. This addition is accomplished by the same transformer which satisfies equation (17c). The sum of these terms is the voltage across a resistor whose value is $\Delta t/Gh\Delta s$ and through which a current equal to F_{ts} flows.

Two observations can be made concerning equations (17) and the resulting circuit. If r does not depend on s, the equations are those of a cylindrical shell, and r may be removed from inside the difference operators. Hence the equations of a noncylindrical shell have the same form as the equations of a cylindrical shell if rotation about the axis u_t/r (rather than tangential displacement) and torque about the axis rF_{st} (rather than tangential force) are used as variables. The rotation and torque about the axis are the natural variables to use in deriving the equations of a circular noncylindrical shell.

Equations (17) also apply with slight modification to the skin and stringers of a noncircular cylindrical shell. In this case r depends on t rather than on s, but this dependency on t will not enter into the derivation of the equations for the skin and stringers. Hence for a noncircular cylindrical shell, r can be eliminated from equation (17e) and can be divided out of equation (17b) to give

$$\Delta_s(F_{st}) - \Delta\phi f_t = 0 \tag{18}$$

Since f_t is the external tangential force per radian of ϕ, $\Delta\phi f_t$ is the total external tangential force per bay in the t-direction.

In figure 6 parts of the u_t circuit corresponding to different values of t are not shown connected. The interconnection is accomplished by means of the currents $\Delta t f_t$ which are the reactions of the rings. The voltages at the points where these currents are inserted are constrained to be equal to the corresponding values of u_t/r for the rings. A complete circuit for a shell, including the circuits for elastic rings, will be shown later. If the rings are assumed to be rigid, the circuits for the rings become quite simple as will be shown.

DERIVATION OF AN ANALOGY FOR AN ELASTIC RING

An electrical analogy for the bending of a ring in its own plane will be derived in this section. In the discussion of shells supported by rings it is usually assumed that the rings are perfectly free to warp out of their own planes. An electrical analogy for a circular ring deforming perpendicular to its own plane has been derived by Russell (ref. 6), but this effect will not be considered here.

It will also be assumed that the effects of shearing stiffness and axial stiffness of the ring are small so that these effects can be ignored. This assumption is made in order to simplify the discussion; these effects can be included in the electrical analogy if desired. In addition the eccentricity of the ring, that is, the distance between the neutral axis of the ring and the point of attachment to the skin, will be assumed to be zero. This important effect can also be included in the electrical analogy.

An element of ring is shown in figure 7. The displacement quantities to be used in the analysis of the ring are the normal and tangential displacement of the center line and the rotation of the normal to the center line. The independent position variable ϕ is the angle between

a horizontal line and the normal to the center line of the unloaded ring. The external tangential and normal loads per radian of ϕ applied along the center line of the ring are f_t and f_n. Distributed moment loads will not be considered. It will be noted in figure 7 that the radius of curvature is not assumed to be constant.

The loads, internal forces, and displacements of the above described ring satisfy the following six first-order differential equations. The equilibrium equations are:

$$\frac{dF_n}{d\phi} = F_t - f_n \tag{19}$$

$$\frac{dF_t}{d\phi} = -F_n - f_t \tag{20}$$

$$\frac{dM}{d\phi} = -F_n r \tag{21}$$

The stress-strain and strain-displacement equations are:

$$\frac{d\theta}{d\phi} = \frac{Mr}{EI} \tag{22}$$

$$\frac{du_n}{d\phi} = u_t + \theta r \tag{23}$$

$$\frac{du_t}{d\phi} = -u_n \tag{24}$$

These equations will be replaced by the corresponding first-order difference equations. In so doing central difference equations will be used so that the quantities appearing behind the derivative symbols in the above equations are defined at values of ϕ midway between the values of ϕ at which the undifferentiated quantities are defined. For example, equation (19) may be approximated by the following equation:

$$(F_n)_2 - (F_n)_0 = \left[(F_t)_1 - (f_n)_1\right]\Delta\phi \tag{25}$$

The same difference-equation notation can be used for this equation as was used in the previous section:

$$\Delta_\phi F_n = \left(F_t - f_n\right)\Delta\phi \tag{26}$$

Position subscripts are not required in this equation. The quantities u_n, F_t, and M are defined at the same points and these points are midway between the points where θ, u_t, and F_n are defined.

Equations (20) to (24) could be replaced by simple difference equations in the same manner that equation (19) was replaced by equation (26). However, it can be demonstrated that the following difference equations are more accurate. They give exactly correct results for a segment of ring which is rigid, has constant radius of curvature, and is uniformly loaded. In other words, for any complete ring with constant radius of curvature, they give correct results for rigid body displacements and the statically determinant parts of the internal forces. These statements require lengthy proofs and instead of the proofs being given they will be partially demonstrated later by means of an example.

The equilibrium equations are:

$$\Delta_\phi F_n = \left(F_t - f_n\right)\left(2 \sin \frac{\Delta\phi}{2}\right) \tag{27}$$

$$\Delta_\phi F_t = -\left(F_n + f_t\right)\left(2 \sin \frac{\Delta\phi}{2}\right) \tag{28}$$

$$\Delta_\phi M = -F_n\left(2r \sin \frac{\Delta\phi}{2}\right) \tag{29}$$

The stress-strain and strain-displacement equations are:

$$\Delta_\phi \theta = \frac{r\Delta\phi}{EI} M \tag{30}$$

$$\Delta_\phi u_n = \left(u_t + r\theta\right)\left(2 \sin \frac{\Delta\phi}{2}\right) \tag{31}$$

$$\Delta_\phi u_t = -u_n\left(2 \sin \frac{\Delta\phi}{2}\right) \tag{32}$$

For small values of $\Delta\phi$, $2 \sin (\Delta\phi/2)$ is approximately equal to $\Delta\phi$ and the above equations approach the corresponding simple central-difference equations. The chord subtending the arc $\Delta\phi$ is $r\left(2 \sin \frac{\Delta\phi}{2}\right)$.

An electrical circuit which identically satisfies the above equations is shown in figure 8. In this circuit displacement quantities are voltages to ground and the loads and internal forces are currents. Equation (29) is satisfied by the currents entering a junction in the upper circuit of figure 8. Equations (27) and (28) are similarly satisfied by the currents in the middle and lower circuits of figure 8. Ideal transformers are used to produce currents flowing into the junctions of one circuit that are proportional to the currents flowing in the "main line" of another circuit.

Equation (30) expresses Ohm's law for the drop in voltage between successive nodes of the upper circuit. Equation (31) is satisfied in the main line of the middle circuit by means of ideal transformer coils which insert voltages in the line proportional to the voltages to ground in the other two circuits. Equation (32) is similarly satisfied in the lower circuit. Each transformer is instrumental in the satisfaction of two equations, an equilibrium equation and a strain-displacement equation. The circuit of figure 8 employs three transformers per cell. If the radius of curvature of the ring is constant (fig. 9(a)), or if the ring can be divided into segments containing several cells for each of which the radius of curvature is constant, a circuit requiring only two transformers per cell can be used. Since r is now assumed to be constant, equations (27) to (32) can be rewritten as follows (where $2 \sin \Delta\phi/2$ has been abbreviated by $\Delta\phi'$):

$$\Delta_\phi F_n = \left(F_t r\right)\frac{\Delta\phi'}{r} - f_n\Delta\phi' \tag{27a}$$

$$\Delta_\phi\left(F_t r\right) = -F_n\left(r\Delta\phi'\right) + \left(f_t r\right)\Delta\phi' \tag{28a}$$

$$\Delta_\phi M = -F_n\left(r\Delta\phi'\right) \tag{29a}$$

$$\Delta_\phi \theta = \frac{r\Delta\phi}{EI} M \tag{30a}$$

$$\Delta_\phi u_n = \left[\frac{u_t}{r} - (-\theta)\right] r\Delta\phi' \tag{31a}$$

$$\Delta_\phi\left(\frac{u_t}{r}\right) = -u_n\left(\frac{\Delta\phi'}{r}\right) \tag{32a}$$

In this form of the equations u_t/r and $F_t r$ replace u_t and F_t as variables as in the case of the skin and stringers for a noncylindrical shell. The purpose of this manipulation has been to get equation (31a) into the form shown, where the increment in u_n is proportional to the difference of the other two displacement quantities. This equation can be satisfied by a single transformer whereas equation (31) required two transformers. A circuit satisfying equations (27a) to (32a) is shown in figure 9(b). In this figure the transformer coils connecting the upper and middle circuits are coupled to the transformer coils in the main line of the lower circuit. This remote coupling is indicated by circled numbers. The currents corresponding to the loads are not shown in this circuit.

CIRCUITS FOR RIGID RINGS

It is frequently possible to assume that some or all of the rings supporting a shell are rigid in their own planes without serious error in the analysis of the shell. This assumption greatly simplifies analytical solutions of shell problems and sometimes eliminates a great deal of the equipment required in an analog-computer solution. Since it is assumed here that only the tangential displacement of the ring is important in the analysis of shells, this is the only coordinate that need be represented at points around the periphery of a rigid ring.

The position of a rigid bulkhead is determined by the displacement of one of its points in two perpendicular directions and by the rotation about an axis perpendicular to its plane (as shown in fig. 10(a)). The tangential displacement at points on the periphery can be computed from these three quantities

$$u_t = Y \sin\phi + Z \cos\phi - \theta r \sin\alpha \tag{33}$$

In an analytical solution Y, Z, and θ are unknown quantities. They are usually regarded as Lagrangian multipliers and equation (33) is regarded as an equation of constraint. In an electrical analogy this

equation of constraint can be satisfied by a network of transformers, a general form of which is shown in figure 10(b). This network also satisfies the equilibrium equations of the rigid ring. Applied loads in the y- and z-directions and applied torque are inserted as currents into the network as shown. Since, in general, this circuit requires three transformers for each tangential displacement it has no advantage over an elastic ring circuit in which the resistors corresponding to the bending stiffness of the ring are set equal to zero. However, in practice, one or more of the terms in equation (33) may be equal to zero because either the unknown or the coefficient may vanish. In such cases the transformer network may be quite simple.

For example, in a shell with a vertical plane of symmetry, loaded symmetrically with respect to this plane,

$$u_t = Z \cos \phi \tag{34}$$

This equation requires one transformer for each value of u_t. It is possible to introduce further simplification by replacing u_t by a variable which depends on ϕ. Let

$$u_t = \bar{u}_t \cos \phi \tag{35}$$

then

$$\bar{u}_t = Z \tag{36}$$

If this change of variable is now introduced into the equations of the skin and stringers (eq. (17)), it will be found that the form of the equations will remain unchanged and that the only effect will be a change in the turns ratio of the transformers coupling the u_s and u_t circuits. Equation (36) is then satisfied by connecting together all the tangential displacement nodes in any one ring.

As another example consider a rigid ring (or bulkhead) which is very thin in one direction (see fig. 11(a)). In this case it may be assumed that the angle between the y-axis and the top and bottom surfaces of the shell is small everywhere except at the ends, where there are closing vertical segments. Furthermore it is usual to assume that Y, the displacement parallel to the long direction, is zero. In this case

$$\left.\begin{aligned} u_t &= -\theta r \sin\alpha && \text{top and bottom surfaces} \\ u_{t_1} &= Z - (\theta c/2) && \text{at left end} \\ u_{t_2} &= -Z - (\theta c/2) && \text{at right end} \end{aligned}\right\} \tag{37}$$

Here again scale factors can be introduced to simplify the equations:

$$\left.\begin{aligned} \bar{u}_t &= -\frac{u_t}{r \sin\alpha} = \theta && \text{top and bottom surfaces} \\ \bar{u}_{t_1} &= u_{t_1} = Z - (\theta c/2) && \text{at left end} \\ \bar{u}_{t_2} &= -u_{t_2} = Z + (\theta c/2) && \text{at right end} \end{aligned}\right\} \tag{38}$$

These equations are satisfied by the simple network of figure 11(b).

With this circuit for a rigid bulkhead and the circuit for the stringers and spars (fig. 6), two spar box wings with unsymmetrical top or bottom surfaces can be analyzed. The extension to multispar wings is simple and direct.

SOLUTION OF PROBLEMS

The Cal-Tech analog computer (ref. 7) was used for the solution of two problems in connection with the preparation of this paper. This computer consists essentially of a storehouse of electrical parts which contains, among other things, the resistors and high-quality transformers required in the solution of stress-analysis problems.

The first problem was the analysis of the simple circular ring shown in figure 9(a) subjected to two opposing concentrated radial loads. Because of the symmetry of the ring and its loads a quadrant of the ring can be substituted for the whole if proper boundary conditions are applied at the ends of the quadrant. At $\phi = 0$ the proper boundary conditions are that θ, u_t, and F_n equal zero. At $\phi = \pi/2$ the proper boundary conditions are that θ and u_t equal zero and that F_n equals P, one-half of the applied load.

In figure 9(b), the quadrant of ring has been represented by a circuit containing four cells. The boundary conditions have been satisfied by setting the corresponding electrical quantities equal to zero at the two ends.

The results of the analysis are presented in table I in dimensionless form. An exact analysis of the problem using differential equations is compared with the analog-computer solution. In addition an exact solution of the difference equations governing the electric circuit is shown. It will be seen that the differential-equation solution and the exact solu- of the difference equations give identical results for the internal shear and internal axial force. This was to be expected since these quantities are statically determinate in the problem investigated. The other quantities show errors due to finite-difference approximation. A comparison of the difference-equation solution and the analog-computer solution shows errors in the computer solution of the order of 1 or 2 percent, which is fairly typical of the results customarily obtained with the Cal-Tech analog computer.

The second problem was the analysis of a conical shell supported by circular elastic rings. As such it provides an example illustrating the manner in which the shell and ring circuits are interconnected. The structure, which has 3 elastic rings and 14 stringers, is shown in figure 12(a), and specifications for the structure are given in table II. This structure is supposed to resemble the aft portion of an aircraft fuselage. The structural weight is divided approximately equally between the skin, stringers, and rings. The rings are somewhat stiffer than those employed customarily in fuselage construction. The number of stringers and the number of rings in the structure are much fewer than the number that would be employed in an aircraft fuselage, so that each stringer of the structure represents several stringers in the fuselage, and the stiffness of intermediate fuselage rings is included in the stiffness of the three main rings shown.

The shell is subjected to symmetrical concentrated vertical loads applied to the ring at the small end of the shell and these loads are reacted at the large end, which is built into a rigid wall. Because of the symmetry of the structure and the applied loads, only a quarter of the shell need be considered if appropriate boundary conditions are applied. This part of the shell has been given a two-coordinate numbering system for the identification of points in the structure. For example, point 42 refers to a point at the intersection of ring number 4 and stringer number 2.

The electrical analogy for the structure is also shown in figure 12. This circuit consists of three parts, the u_s and u_t circuits and the ring circuits. In figure 12(b) only one ring circuit is shown since the other two have an identical appearance. The connections between the ring

circuit shown and the u_t circuit are indicated by the circled letters a, b, and c. Currents corresponding to the interaction forces between the skin and the rings flow through these connections. In figure 12(a) a cutout is indicated in the middle bay. Electrical parts corresponding to the cutout (shown dotted in figs. 12(c) and 12(d)) are removed when the cut panel is removed.

The boundary conditions at the vertical plane of symmetry are that the shear stresses in shear panels intersected by this plane are zero and that in the ring circuits θ, u_t, and F_n are all zero except when vertical loads are applied to the ring in the plane of symmetry, in which case F_n is equal to P, one-half of the applied load. The boundary conditions at the horizontal plane of symmetry are that the displacement in the s-direction u_s is zero and that in the ring circuits u_n, M, and F_t are zero except when vertical loads are applied to the ring in the horizontal plane of symmetry, in which case F_t is equal to -P. The boundary conditions at the large end of the shell are that both u_s and u_t are equal to zero. At the small end the boundary conditions are conveniently expressed as the absence of applied loads except as indicated. All of the above boundary conditions have been satisfied in the electrical circuit by means of short and open circuits. Short circuits are used to set displacements equal to zero while open circuits are used to set internal forces equal to zero.

The electrical circuits shown in figure 12 use 27 resistors and 27 transformers. In calculating the values of the transformer turns ratios and resistors to be used in the circuit it is necessary to make use of scale factors. This aspect of the problem has been omitted in the present discussion in order that the electrical quantities shown in the circuit diagram may have a direct significance in terms of mechanical quantities. A brief discussion of the scale-factor methods employed with the Cal-Tech analog computer is given in the appendix of reference 8.

The structural and loading conditions that were investigated in this problem are given in table II. It was desired to investigate the effect of the following things on the distribution of internal forces: The effect of the stiffness of the rings, the effect of the location of the vertical load on the end rings, and the effect of a cutout in the middle bay.

The results of these investigations are given in table II. The quantities recorded according to the numbering system previously discussed are the stringer forces F_s, the panel shears F_{ts}, the ring bending moments M, and the vertical displacements. The vertical displacements have been obtained by vectorially combining the tangential and normal displacements of the rings. The tabulated results are subject to experimental errors and when four significant figures are given the fourth figure is entirely unreliable.

The main conclusion to be drawn from these results is that for the shell tested and the loads which were applied the stiffness of the rings has a rather small effect on the distribution of internal forces. The only change made between conditions (1) and (2) was that in condition (2) the rings were made five times as stiff as in condition (1). It will be seen that the stiffness of the rings has very little effect on the internal forces except between the second and third rings. However, the distortion of the third ring in condition (1), as given by the vertical displacements, is significant. In another condition, the results of which are not tabulated, the first and second rings were stiffened by a factor of 10 while the third ring retained its normal stiffness. The difference in the results between this condition and condition (1) was negligible.

The results of condition (3) indicate that the effect of ring stiffness is considerably less when the applied vertical loads are in a horizontal plane than when they are in a vertical plane. The results of conditions (4) and (5) indicate that even moderately small symmetrical cutouts produce a severe redistribution of the internal forces and that, in this case, the effect of ring stiffness is important if accurate results are desired.

California Institute of Technology,
Pasadena, Calif., November 25, 1953.

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TABLE I.- SOLUTION OF CIRCULAR-RING PROBLEM WITH

FOUR FINITE-DIFFERENCE CELLS PER QUADRANT

Quantity	ϕ (o)	Differential equations	Difference equations	Analog computer
$\frac{F_n}{P}$	22.5 45.0 67.5	0.3827 .7071 .9239	0.3827 .7071 .9239	0.387 .702 .922
$\frac{F_t}{P}$	11.25 33.75 56.25 78.75	0.9808 .8315 .5556 .1951	0.9808 .8315 .5556 .1951	0.982 .824 .566 .210
$\frac{M}{Pr}$	11.25 33.75 56.25 78.75	0.3442 .1948 -.0810 -.4415	0.3401 .1907 -.0851 -.4456	0.340 .192 -.083 -.447
$\frac{Q}{\frac{Pr^2}{EI}}$	22.5 45.0 67.5	0.1327 .2071 .1739	0.1335 .2084 .1750	0.1338 .2064 .1703
$\frac{u_t}{\frac{Pr^3}{EI}}$	22.5 45.0 67.5	0.0487 .0706 .0515	0.0514 .0747 .0548	0.0515 .0756 .0559
$\frac{u_n}{\frac{Pr^3}{EI}}$	11.25 33.75 56.25 78.75	-0.1271 -.0573 .0493 .1349	-0.1318 -.0596 .0509 .1405	-0.1310 -.0586 .0535 .1435

TABLE II.- RESULTS FROM STUDY OF CONICAL SHELL

[Specifications for shell: Number of stringers, 14; cross-section area of stringers, 0.60 sq in.; thickness of skin, 0.030 in.; Young's modulus, 10.4×10^6 psi; and shear modulus, 4.0×10^6 psi. P = 1 kip.]

Panel point	Condition				
	(1) (a)	(2) (b)	(3) (c)	(4) (d)	(5) (e)
	Stringer forces, F_s, kips				
2	-0.945	-0.945	-0.942	-0.330	-0.577
4	-1.708	-1.708	-1.708	-1.550	-1.632
6	-2.118	-2.050	-2.124	-2.475	-2.300
22	-.688	-.711	-.722	.767	.688
24	-1.282	-1.282	-1.291	-1.409	-1.321
26	-1.612	-1.594	-1.587	-2.124	-2.180
42	-.308	-.380	-.452	-1.783	-1.789
44	-.676	-.714	-.745	-.536	-.650
46	-1.002	-.935	-.882	-.445	-.326
	Panel shears, F_{ts}, kips				
11	1.206	1.195	1.191	1.593	1.726
13	.954	.956	.963	.509	.469
15	.513	.531	.536	.372	.155
31	1.596	1.550	1.522	0	0
33	1.217	1.231	1.254	2.537	2.457
35	.618	.661	.707	1.687	1.838
51	1.985	2.046	2.080	2.740	2.789
53	1.685	1.662	1.632	.994	1.032
55	1.013	.937	.880	.449	.334
	Ring bending moments, M, in-kips				
22	-0.07	-0.03	0.02	3.58	4.08
24	-.03	-.02	.01	.96	1.28
26	.06	.02	-.02	-2.40	-2.84
42	.20	.05	-.08	-3.72	-3.88
44	.10	.01	-.02	-1.08	-1.28
46	-.18	-.07	.06	2.48	2.64
62	4.45	4.36	-2.28	5.53	5.45
64	3.84	3.52	-.56	4.12	3.88
66	-5.25	-5.65	1.44	-5.93	-6.45
	Vertical displacements, in.				
22	0.0288	0.0283	0.0282	0.0133	0.0247
24	.0290	.0286	.0288	.0262	.0282
26	.0283	.0283	.0283	.0592	.0336
42	.0875	.0871	.0876	.2180	.1887
44	.0880	.0872	.0870	.2105	.1870
46	.0895	.0885	.0880	.1912	.1830
62	.1577	.1670	.1735	.2595	.2515
64	.1660	.1701	.1702	.2660	.2530
66	.1926	.1770	.1627	.3005	.2605

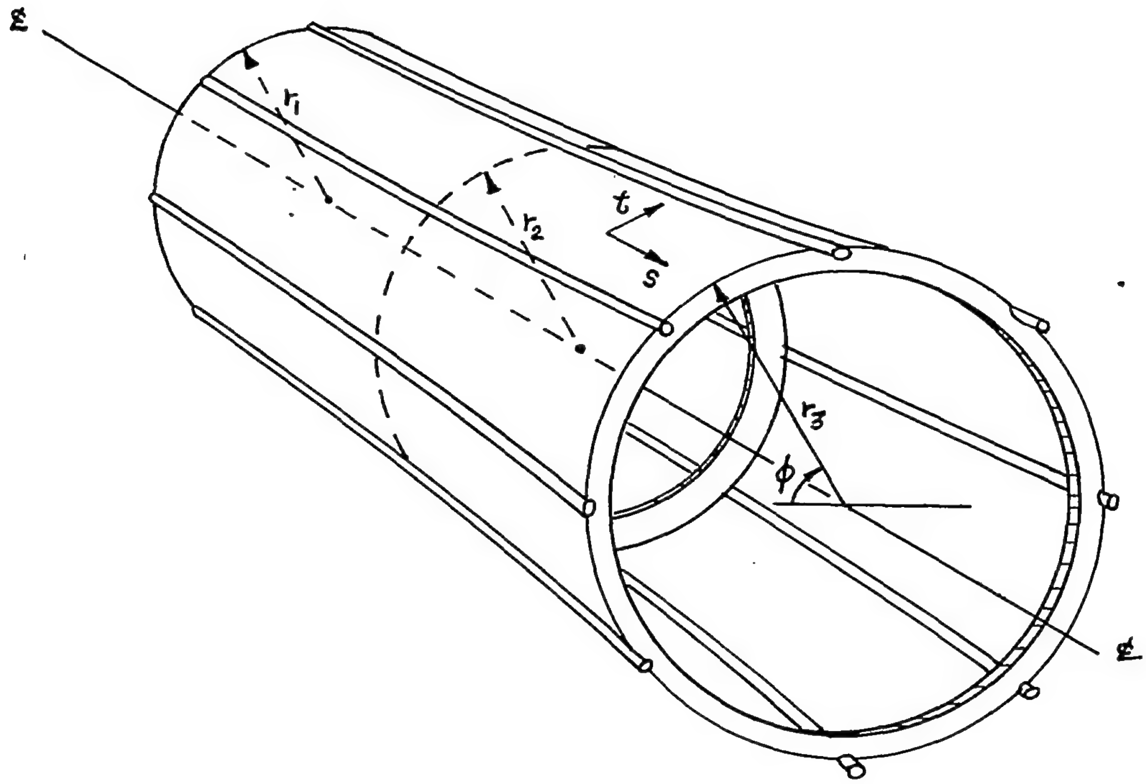
[a]Condition (1): Vertical loads applied to points A and A' of end ring (see fig. 12(a)). No cutouts. Moment of inertia of rings, 5.33 in.^{4}.

[b]Condition (2): Same as condition (1) except moment of inertia of rings increased by a factor of 5.

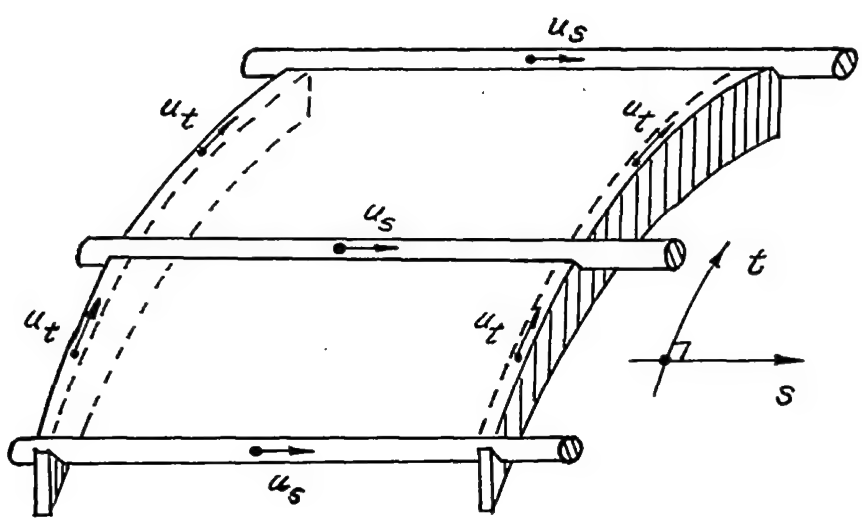
[c]Condition (3): Same as condition (1) except vertical loads applied to points B and B' (see fig. 12(a)).

[d]Condition (4): Same as condition (1) except with symmetrical cutouts in center bay (see fig. 12(a)).

[e]Condition (5): Same as condition (4) except moment of inertia of rings increased by a factor of 5.



(a) Segment of shell showing orthogonal coordinates.



(b) Enlarged portion of shell showing displacements.

Figure 1.- Circular noncylindrical stiffened shell.

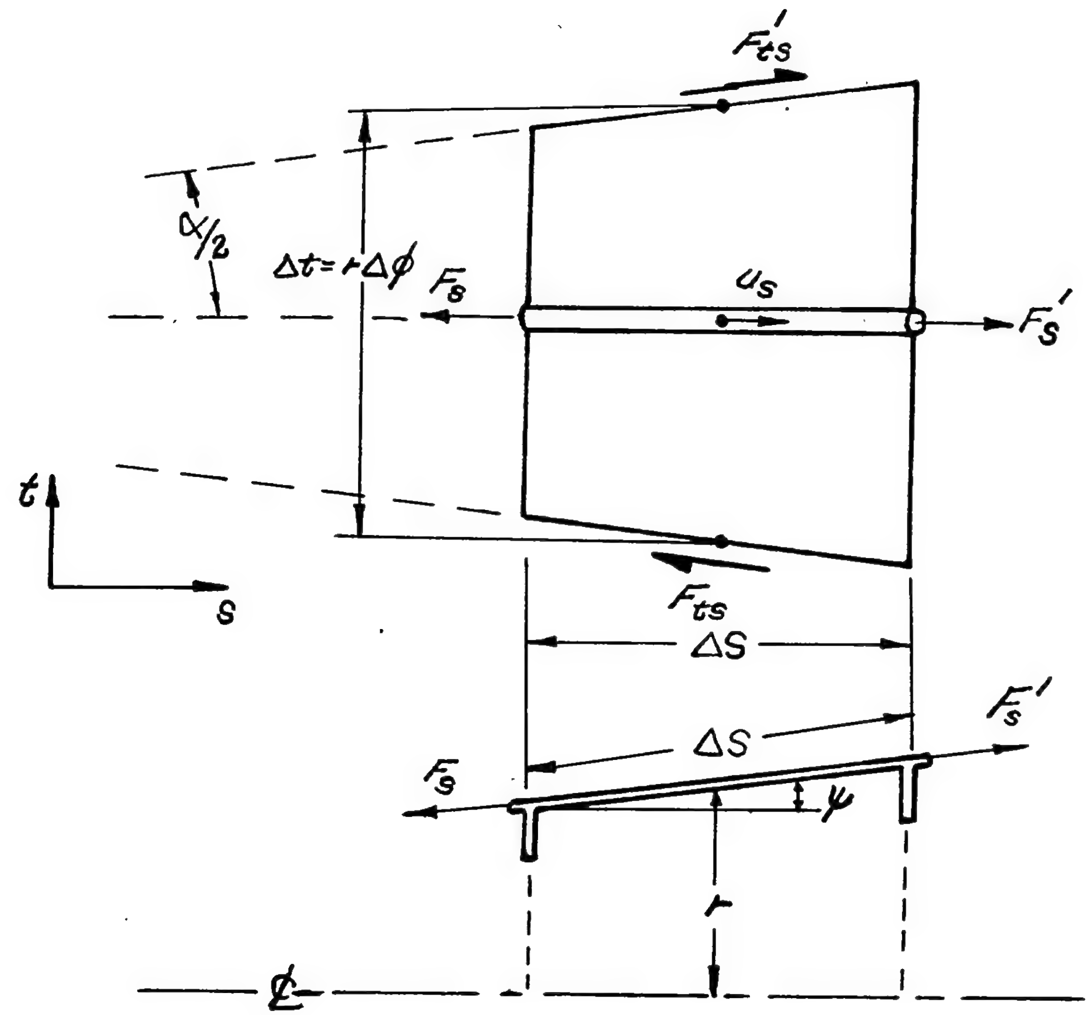


Figure 2.- Equilibrium of portion of shell between two adjacent rings with center on a stringer.

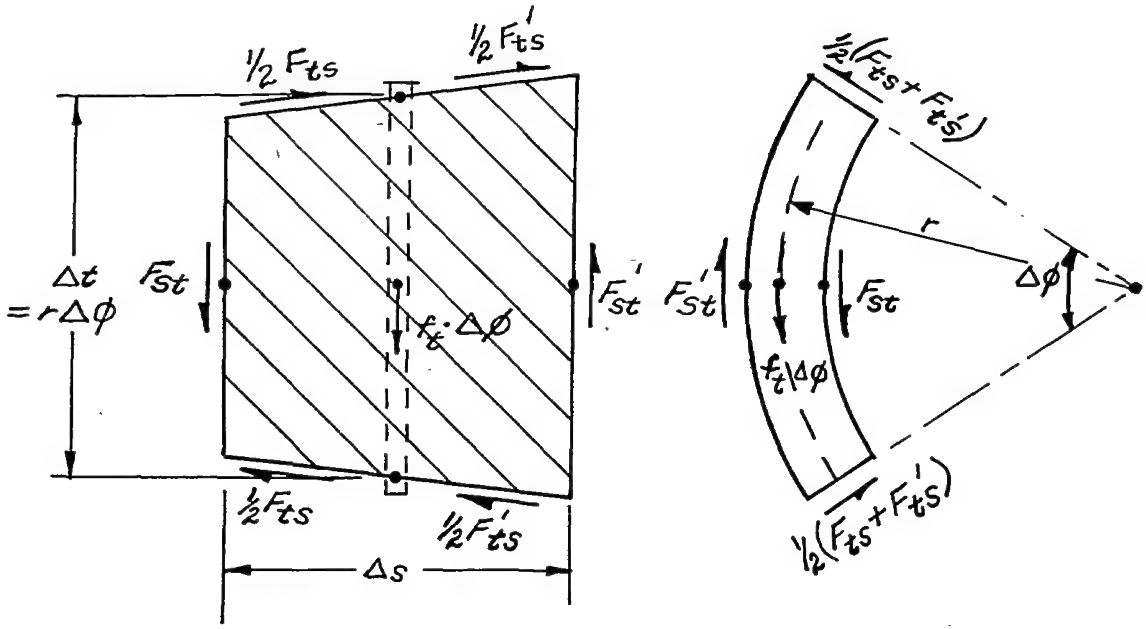


Figure 3.- Equilibrium of portion of shell between two adjacent stringers with center on a ring.

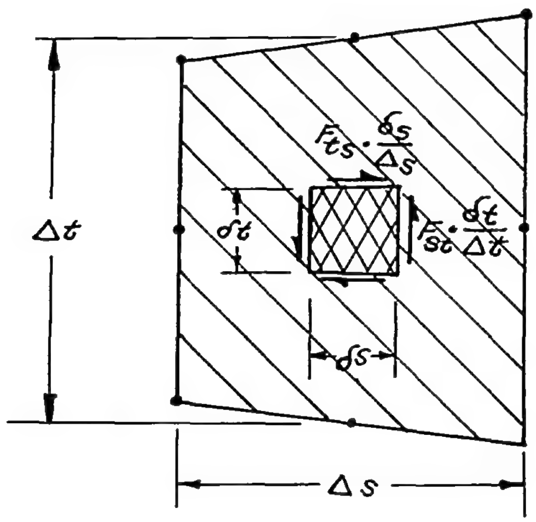


Figure 4.- Equilibrium of shearing forces in a panel.

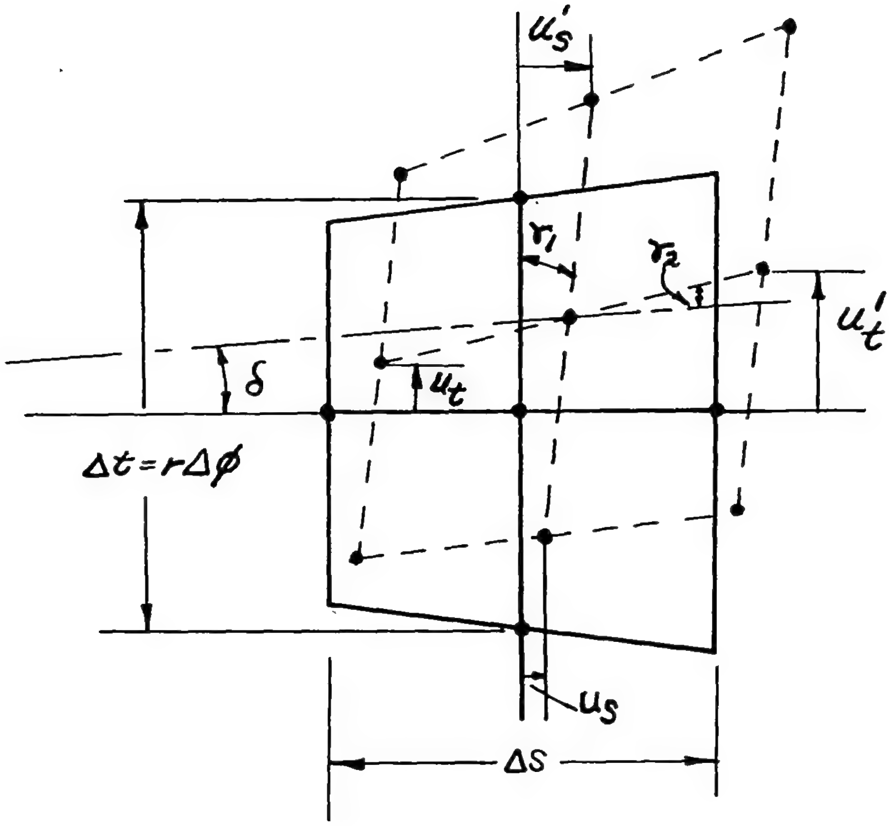
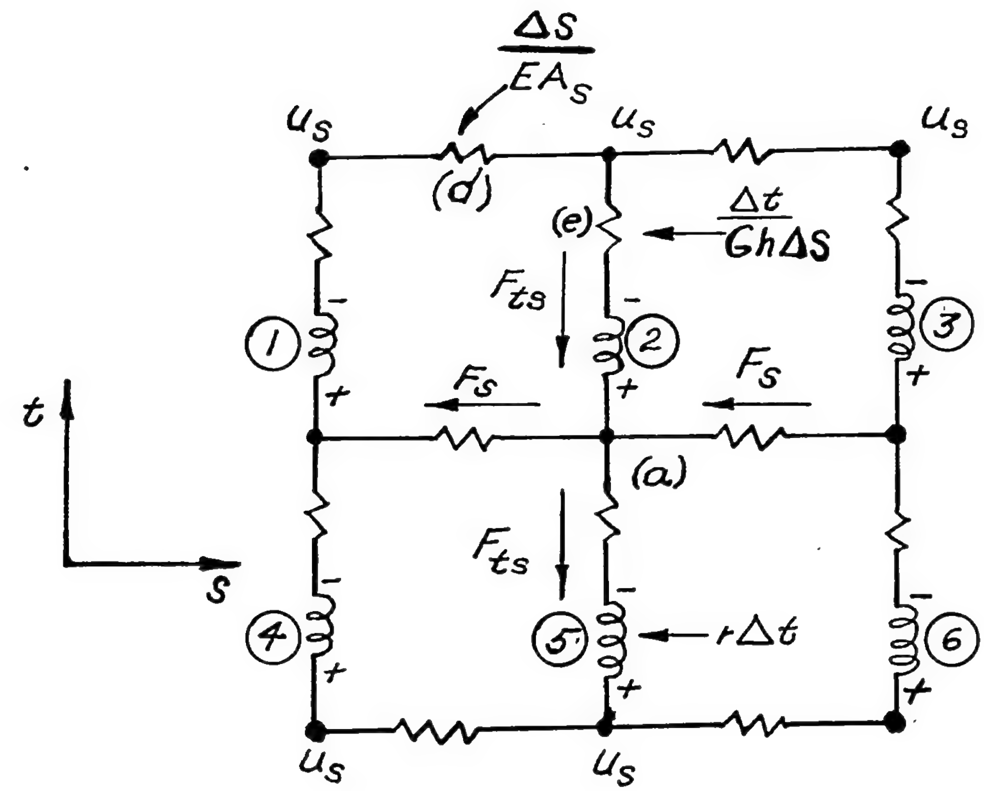
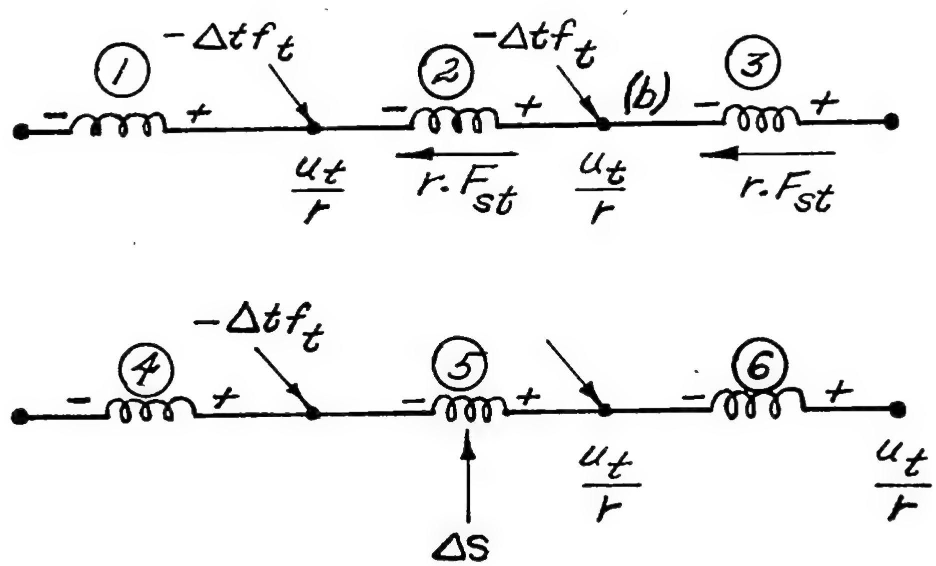


Figure 5.- Shear strain of a panel.

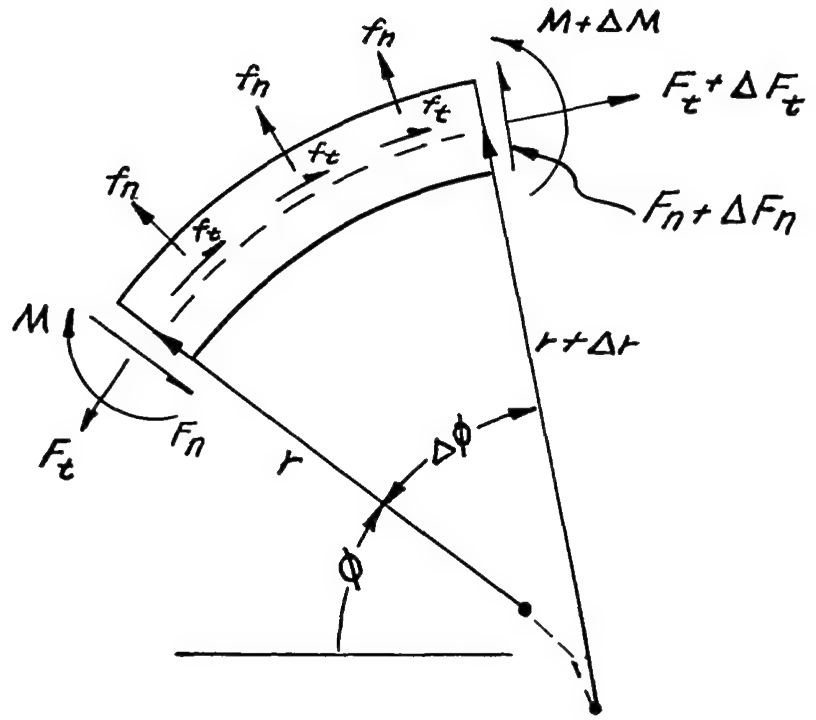


(a) u_s circuit.

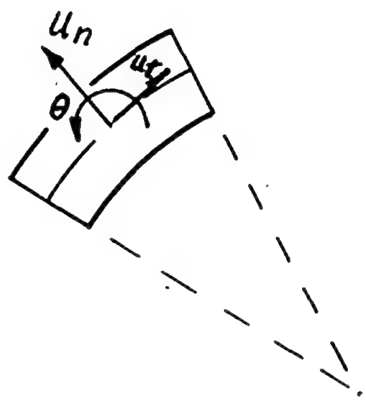


(b) u_t circuit.

Figure 6.- Electrical analogy for stringers and skin of a circular noncylindrical shell.



(a) Forces.



(b) Displacements.

Figure 7.- Segment of a ring showing applied loads, internal forces, and displacements.

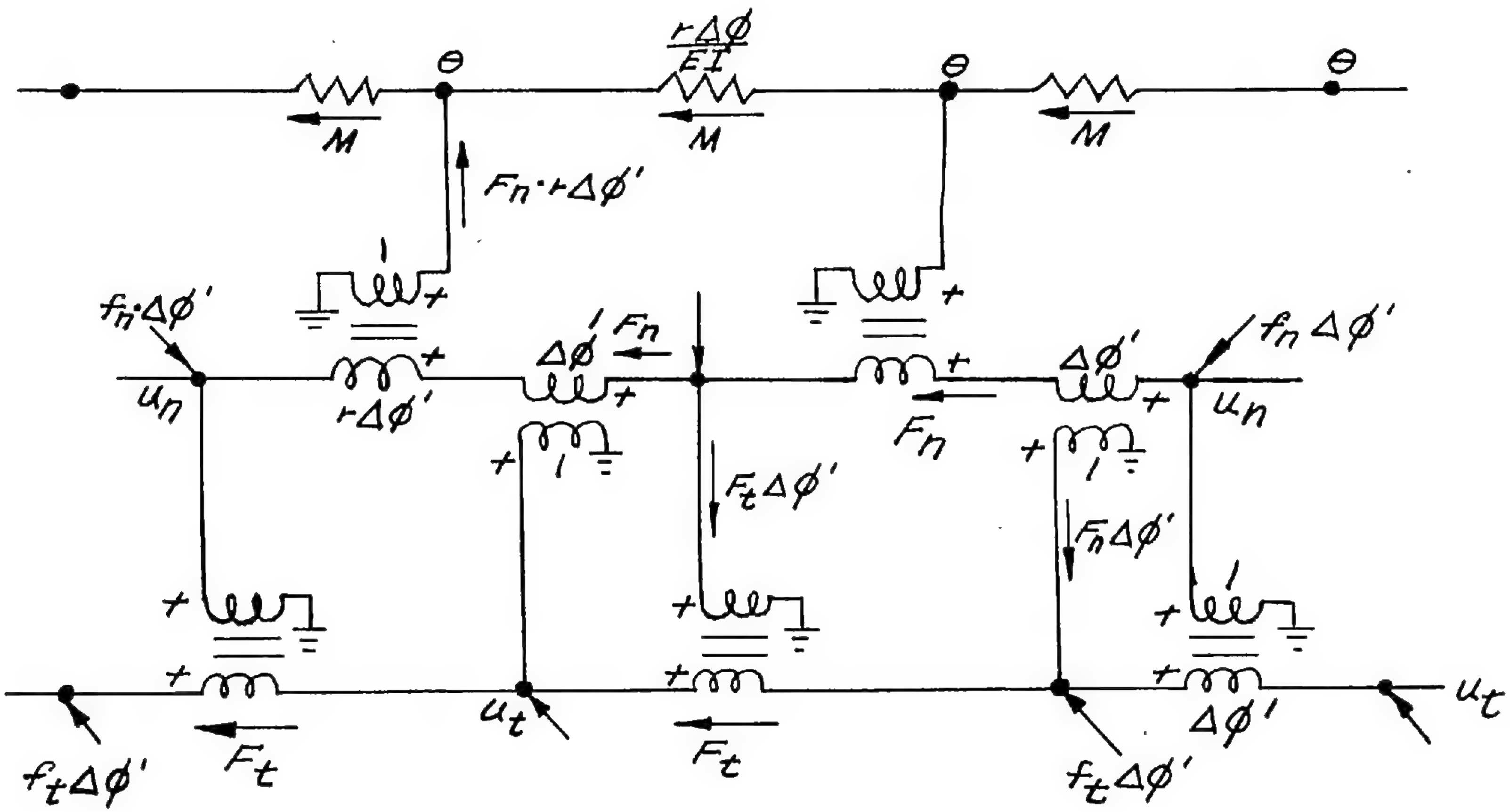
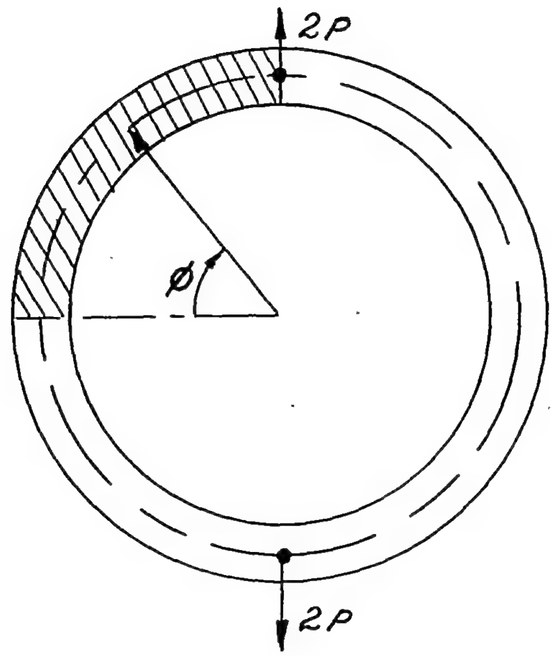
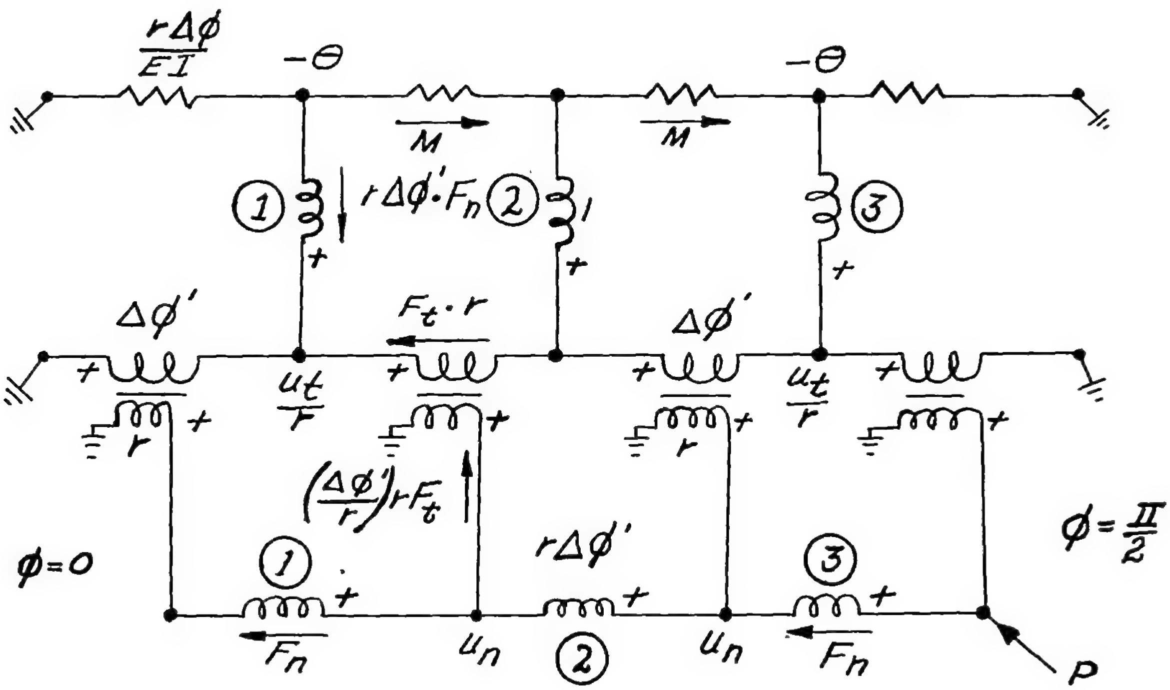


Figure 8.- Electrical analogy for a ring with variable radius of curvature. $\Delta\phi' = 2 \sin \frac{\Delta\phi}{2}$.

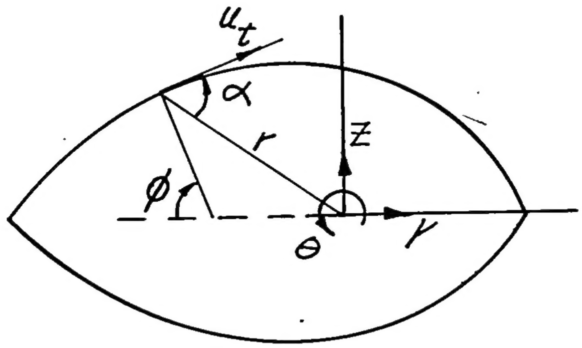


(a) Ring and loads.

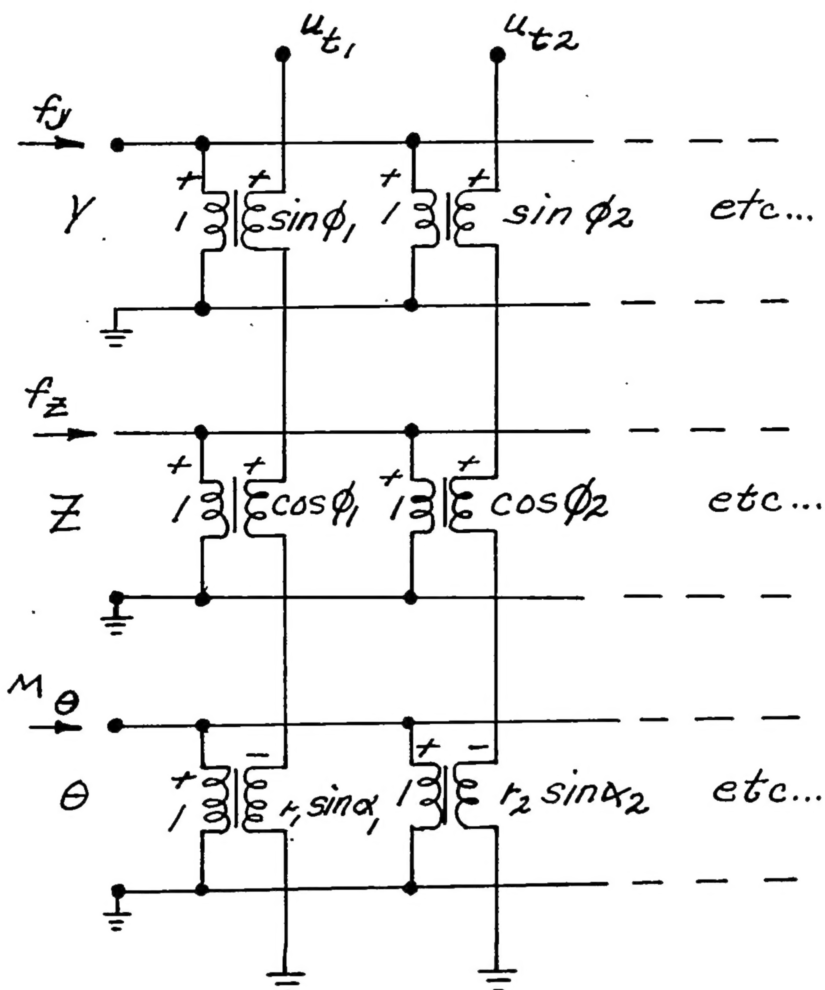


(b) Analog circuit for quadrant of ring.

Figure 9.- Electrical analogy for a circular ring subjected to concentrated loads.

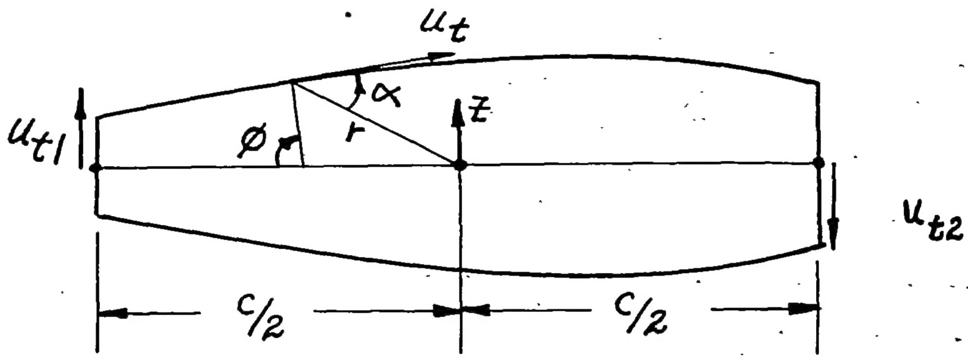


(a) Rigid bulkhead. $u_t = Z \cos \phi + Y \sin \phi - \theta r \sin \alpha$.

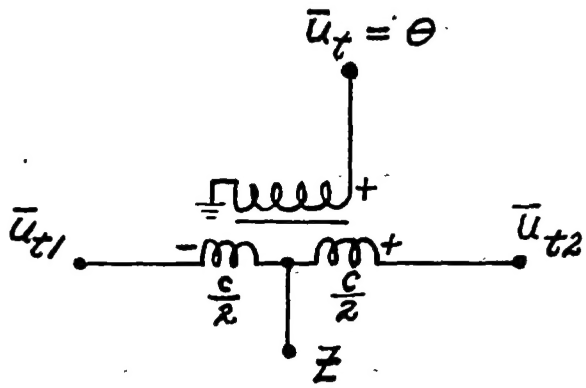


(b) Analogous electric circuit.

Figure 10.- Electrical analogy for a rigid bulkhead.

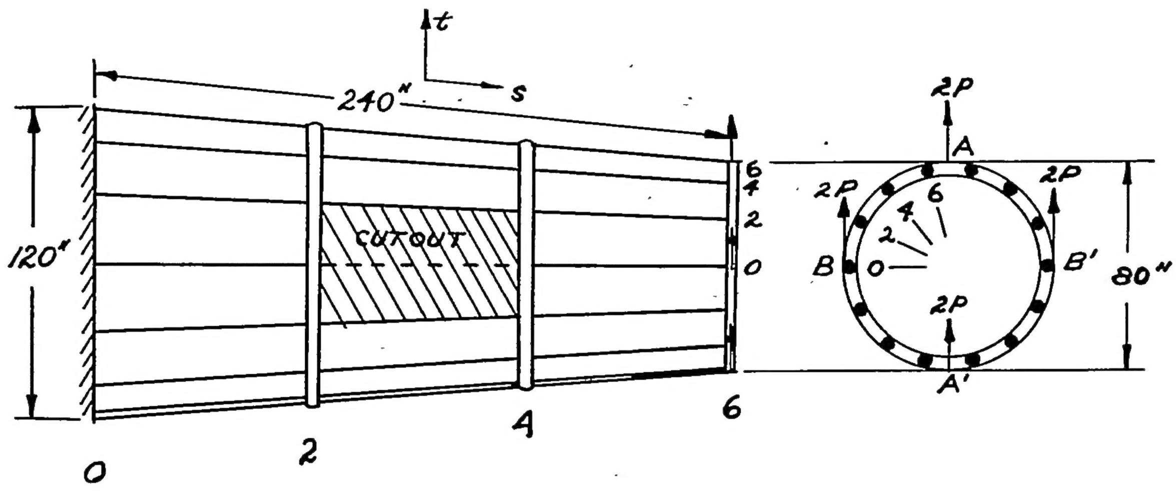


(a) Flat rigid bulkhead.

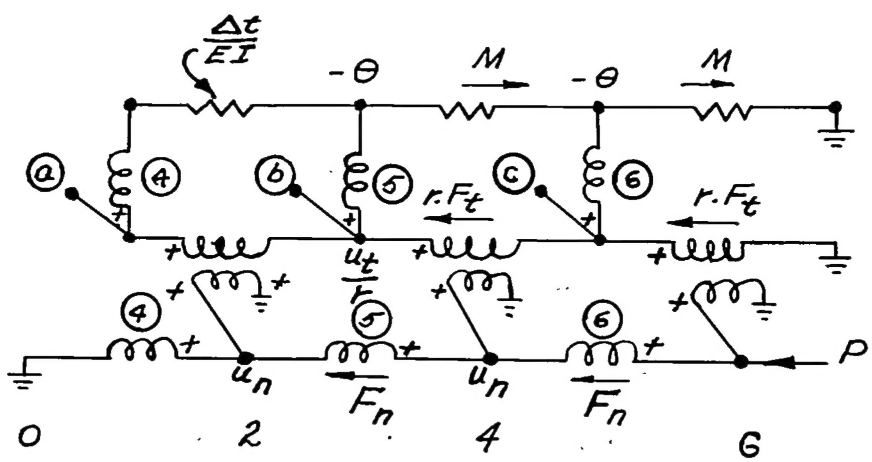


(b) Analogous electric circuit.

Figure 11.- Electrical analogy for a flat rigid bulkhead.

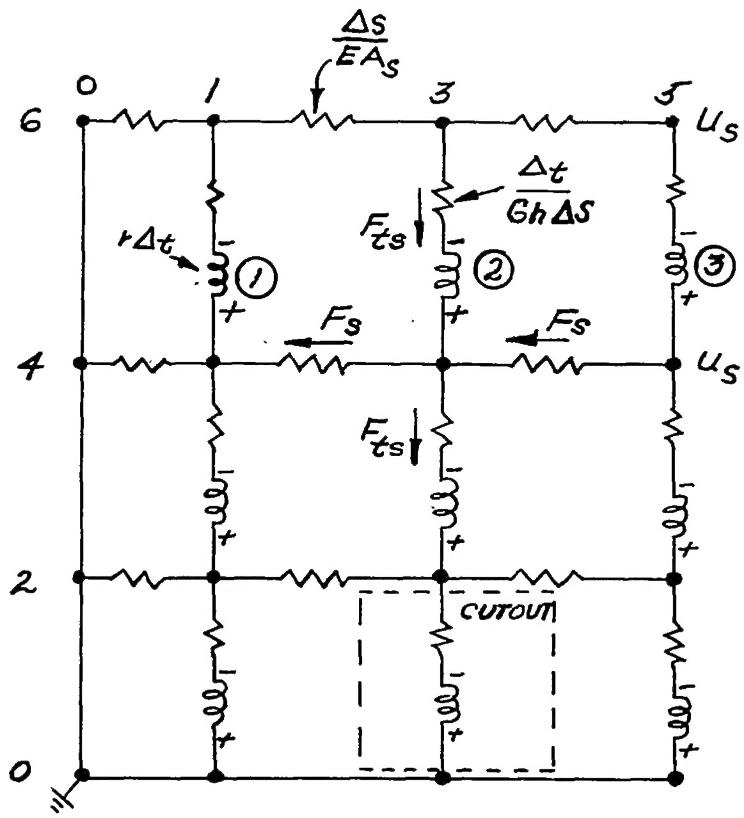


(a) Conical shell.

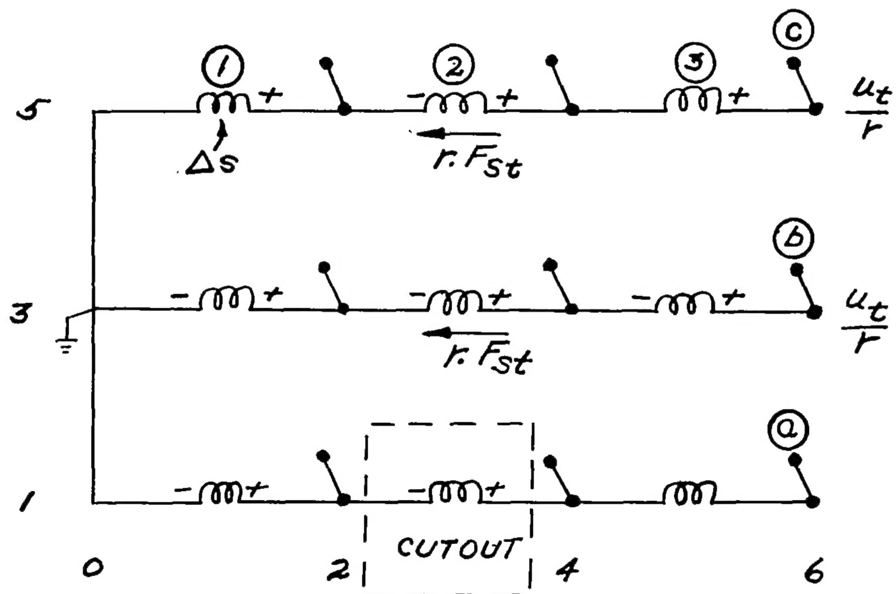


(b) Typical ring circuit (3 required).

Figure 12.- Electrical analogy for conical-shell problem.



(c) u_s circuit.



(d) u_t circuit.

Figure 12.—Concluded.

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